ELASTO-PLASTIC PROBLEM FOR A SHEET WEAKENED BY A DOUBLY PERIODIC SYSTEM OF CIRCULAR HOLES

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The problem of determination of boundaries separating the elastic and plastic regions of an infinite perforated sheet with a triangular lattice of circular holes (Fig. 1) is examined below for the case when on the contour of each hole the uniformly distributed stress σ_0 is given. In the solution of the problem it is assumed that the level of stress and the spacing of the lattice are such that the boundaries of the hole are completely engulfed by the corresponding plastic region but at the same time neighboring plastic regions do not intersect.

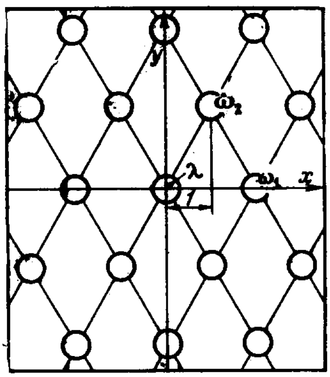

Fig. 1

The problem of elastic equilibrium of the sheet weakened by a doubly periodic system of holes was examined in a series of papers [1 and 2]. With increasing stress in such a sheet plastic regions arise around the holes. The distribution of these regions has also a doubly periodic character

The problem of development of a plastic region around a single hole with determination of the boundary of separation of the elastic and plastic regions was formulated and solved in papers of Nadai [3] and Galin [4]. Later Kosmodamianskii [5] making use of solution [4] for the plastic region, examined a sheet weakened by a series of periodically distributed holes.

For the solution of the problem it is natural to combine the method developed in the solution of doubly periodic elastic problem [2] with the method proposed in [4] for the solution of the problem concerning the boundary of the elastic and plastic regions in the case of a single hole in the sheet.

Let there exist a doubly periodic triangular lattice with circular holes having the radius λ ($\lambda < 1$) and centers at points

$$p_{mn} = m\omega_1 + n\omega_2 \qquad (m,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots), \qquad \omega_1 = 2, \qquad \omega_2 = 2e^{1/3 i\pi}$$

Let us designate the contour of the hole with the center at point p_{mn} by L_{mn}, the boundary of the corresponding plastic zone by Γ_{mn} and the exterior of contour Γ_{mn} by D_z.

The condition of plasticity is taken in the form

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k_0^2$$

On the contour of the hole L_{mn} the boundary conditions have the form

$$\sigma_r = -\sigma_0, \qquad \tau_{r\theta} = 0$$

Stress functions of Kolosov-Muskhelishvili [6] for the plastic region have the form [4]

$$\Phi_1^*(z)=\frac{k_0-\sigma_0}{2}+k_0\ln\frac{z}{\lambda}, \qquad \Psi_1^*(z)=0.$$

In the elastic regions we have for the stress functions $\Phi_2^*(z)$ and $\Psi_2^*(z)$

$$\sigma_0+\sigma_r=4\,\mathrm{Re}\,\Phi_2^*(z), \qquad \tau_{r0}=0, \qquad \sigma_0-\sigma_r+2i\tau_{r0}=2\left[\bar{z}\Phi_2^{*\prime}(z)+\Psi_2^*(z)\right]e^{2i\theta}$$

Combining the solutions for the elastic and plastic regions on the boundary of their separation we obtain the boundary conditions for the elastic region. On the contour Γ_{00} the following conditions must be satisfied

$$\mathrm{Re}\,\Phi_2^*(z)=\mathrm{Re}\Phi_1^*(z), \qquad \sigma_0-\sigma_r=2k_0$$

or

$$\mathrm{Re}\Phi_2^*(z)=\frac{k_0-\sigma_0}{2}+k_0\ln\frac{|z|}{\lambda}; \qquad \bar{z}\Phi_2^{*\prime}(z)+\Psi_2^*(z)=k_0\frac{\bar{z}}{z}$$

Let function $z=w(\zeta)$ produce conformal representation of region D_z on region D_ζ in the plane ζ which is the exterior of circumference γ_{mn} of radius μ with centers at points p_{mn}. The boundary conditions on γ_{00} are written in the form

$$\mathrm{Re}\Phi_2(\zeta)=\frac{k_0-\sigma_0}{2}+\frac{k_0}{2}\left[\ln\frac{w(\zeta)}{\lambda}+\ln\frac{\overline{w(\zeta)}}{\lambda}\right] \tag{1}$$

$$\frac{\overline{w(\zeta)}}{w'(\zeta)}\Phi_2'(\zeta)+\Psi_2(\zeta)=k_0\frac{\overline{w(\zeta)}}{w(\zeta)} \tag{2}$$

where

$$\Phi_2(\zeta)=\Phi_2^*[w(\zeta)], \qquad \Psi_2(\zeta)=\Psi_2^*[w(\zeta)]$$

Expression (2) can be transformed in the following manner. From (1) it follows that in D_ζ

$$\Phi_2(\zeta)=\frac{k_0-\sigma_0}{2}+k_0\ln\frac{w(\zeta)}{\lambda}-k_0\ln\frac{\zeta}{\mu} \tag{3}$$

Substituting (3) into (2) we obtain the second boundary condition on γ_{00} in the form

$$\zeta w'(\zeta)\Psi_2(\zeta)=-k_0\overline{w(\zeta)} \tag{4}$$

The solution in the plane ζ will be sought in the form of series [2]

$$\Phi_2(\zeta)=\alpha_0+\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\mu^{2k+2}\wp^{(2k)}(\zeta)}{(2k+1)!} \tag{5}$$

$$\Upsilon_2(\zeta)=\beta_0+\sum_{k=0}^{\infty}\beta_{2k+2}\frac{\mu^{2k+2}\wp^{(2k)}(\zeta)}{(2k+1)!}-\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\mu^{2k+2}Q^{(2k+1)}(\zeta)}{(2k+1)!} \tag{6}$$

where $\wp(\zeta)$ is the elliptical function of Weierstrass, $Q(\zeta)$ is a special meromorphic function

$$\wp(z)=\frac{1}{z^2}+\sum_{m,n}{}'\left\{\frac{1}{(z-p)^2}-\frac{1}{p^2}\right\}, \qquad Q(z)=\sum_{m,n}{}'\left\{\frac{\bar{p}}{(z-p)^2}-2z\frac{\bar{p}}{p^3}-\frac{\bar{p}}{p^2}\right\}$$

$$p=m\omega_1+n\omega_2, \qquad (m,\ n=0,\ \pm1,\ \pm2,\ \ldots)$$

The representation for $w(\zeta)$ follows from boundary condition (1) and the requirement of preservation of poles p_{mn} in mapping:

$$w(\zeta)=\zeta+\sum_{k=0}^{\infty}A_{2k+2}\frac{\mu^{2k+2}\wp^{(2k-1)}(\zeta)}{(2k+1)!} \tag{7}$$

Now we shall put together relationships which must be satisfied by coefficients of Expressions (5) to (7). The condition of equality to zero of the principal vector of forces acting on the arc which connects two congruent points in D_ζ, is equivalent to Eqs.

$$g(\zeta+\omega j)-g(\zeta)=0 \qquad (j=1,\ 2)$$

$$g(\zeta)=\varphi(\zeta)+\zeta\overline{\Phi(\zeta)}+\psi(\zeta), \qquad \varphi'(\zeta)=\Phi(\zeta), \qquad \psi'(\zeta)=\Psi(\zeta)$$

and leads to relationships [2]

$$\alpha_0 = {}^1/_3 \sqrt{3}\, \pi\beta_2\mu^2, \qquad \beta_0 = {}^1/_6 \sqrt{3}\, \pi\alpha_2\mu^2 \tag{8}$$

Taking into account that $\sigma_1 = \sigma_2 = -\sigma_0$ we have $\alpha_2 = 0$, $\beta_2 \neq 0$.

Symmetry conditions for the triangular lattice are written in the form

$$\Phi_2(\zeta e^{1/_3 i\pi}) = \Phi_2(\zeta), \qquad \Psi_2(\zeta e^{1/_3 i\pi}) = \Psi_2(\zeta)\, e^{-2/_3 i\pi}, \qquad w(\zeta e^{1/_3 i\pi}) = w(\zeta)\, e^{1/_3 i\pi}$$

From this it follows that

$$\alpha_{6k\pm 2} = \beta_{6k+2\pm 2} = A_{6k\pm 2} = 0$$
$$\text{for } k = 0, 1,$$

Further, let us substitute Expressions (5) and (7) into the boundary condition (1), multiply the obtained expression by $\frac{1}{2}\pi i\zeta$ and integrate over the contour γ_{00}. The integrals are readily computed if Eqs. (70.3) and (70.4′) in [6] are used. We obtain

$$\alpha_0 + \sum_{k=1}^{\infty}{}' \alpha_{6k}\mu^{6k} r_{0,3k-1} = \frac{k_0 - \sigma_0}{2} + k_0 \ln\frac{\mu}{\lambda}\left[1 + \sum_{k=1}^{\infty}{}' A_{6k}\mu^{6k} r_{0,3k-1}\right] \tag{9}$$

where

$$r_{0,3k-1} = \sum_{m,n}{}' \frac{1}{p_{mn}^{6k}}$$

For construction of the remaining equations with respect to coefficients of representations of functions $\Phi_2(\zeta)$, $\Psi_2(\zeta)$ and $w(\zeta)$ in (4) to (6) we initially expand these functions in Laurent series in the vicinity of the point $\zeta = 0$:

$$\Phi_2(\zeta) = \alpha_0 + \sum_{k=1}^{\infty}{}' \alpha_{6k}\left(\frac{\mu}{\zeta}\right)^{6k} + \sum_{k=1}^{\infty}{}' \alpha_{6k}\mu^{6k} \sum_{j=0}^{\infty}{}' r_{3j,\,3k-1}\zeta^{6j} \tag{10}$$

$$\Psi_2(\zeta) = \sum_{k=0}^{\infty}{}' \beta_{6k+2}\left(\frac{\mu}{\zeta}\right)^{6k+2} + \sum_{k=0}^{\infty}{}' \beta_{6k+2}\mu^{6k+2}\sum_{j=0}^{\infty}{}' r_{3j+2,\,3k}\zeta^{6j+4} -$$
$$- \sum_{k=0}^{\infty}{}' 6k\alpha_{6k}\mu^{6k} \sum_{j=0}^{\infty}{}' s_{3j+2,\,3k-1}\zeta^{6j+4} \tag{11}$$

$$w(\zeta) = \zeta - \sum_{k=1}^{\infty}{}' \frac{A_{6k}\mu}{6k-1}\left(\frac{\mu}{\zeta}\right)^{6k-1} + \sum_{k=1}^{\infty}{}' A_{6k}\mu^{6k}\sum_{j=0}^{\infty}{}' \frac{r_{3j,\,3k-1}}{6j+1}\zeta^{6j+1} \tag{12}$$

Here

$$r_{jk} = \frac{(2j+2k+1)!\, g_{j+k+1}}{(2j)!\,(2k+1)!\,2^{2j+2k+2}}, \qquad s_{jk} = \frac{(2j+2k+2)!\,\rho_{j+k+1}}{(2j)!\,(2k+1)!\,2^{2j+2k+2}}$$

$$g_j = \sum_{m,n}{}' \frac{1}{T^{2j}}, \qquad \rho_j = \sum_{m,n}{}' \frac{\overline{T}}{T^{2j+1}}, \qquad T = \frac{P_{mn}}{2} = m + ne^{1/_3 i\pi}$$

Substituting Expressions (10) to (12) into boundary conditions (1) and (4) on the contour $\gamma_{00}\{\zeta = \mu e^{i\theta}\}$ and comparing coefficients for $\cos 6k\theta$ $(k = 0, 1, 2,\ldots)$, we obtain an infinite system of nonlinear algebraic equations with respect to α, β and A (it is expedient to differentiate Eq. (1) with respect to θ beforehand). I n the following we present equations of the first approximation.

In the written equations, the terms α_{6j}/k_0, and β_{6j+2}/k_0, are designated by $\alpha_{6'j}$, and $\beta_{6'j+2}$, respectively, and the prime is omitted:

$$2d_1c_0 = c_1, \quad a\beta_2 + A_6\gamma_0 + A_6\beta_8\mu^{12}r_{32} = a$$
$$a\beta_8 + A_6\beta_2 = {}^1/_7 A_6\mu^{12}r_{32}$$
$$a\gamma_0 + A_6\gamma_1 + A_6\beta_2\mu^{12}r_{32} = -{}^1/_5 A_6$$

where

$$d_1 = \alpha_6 (1 + \mu^6 r_{32}), \quad a = 1 + A_6 \mu^6 r_{02},$$
$$c_0 = a^2 + (1/_{25} + 1/_{49} \mu^{24} r_{32}^2) A_6^2$$
$$c_1 = -2aA_6 (1/_5 - 1/_7 \mu^{12} r_{32})$$
$$\gamma_j = \beta_2 r_{3j+2,0} \mu^{6j+6} + \beta_8 r_{3j+2,3} \mu^{6j+12} - 6\alpha_6 s_{3j+2,2} \mu^{6j+10}$$
$$(j = 0, 1)$$

Results of calculations in the first two approximations are given in Table 1

TABLE 1

μ	0.2	0.3	0.4	0.5	0.6	0.7	0.8
	Первое приближение						
A_6	$-15\cdot10^{-5}$	-0.0017	-0.0094	-0.036	-0.102	-0.226	-0.364
α_0	0.018	0.041	0.072	0.113	0.164	0.225	0.302
α_6	$3\cdot10^{-5}$	0.0003	0.002	0.007	0.020	0.042	0.060
β_2	1	1	1	1	1.002	1.011	1.039
β_8	$15\cdot10^{-5}$	0.0017	0.0094	0.036	0.103	0.229	0.379
	Второе приближение						
A_6	$-15\cdot10^{-5}$	-0.0017	-0.0094	-0.036	-0.098	-0.194	
A_{12}	0	0	0	0	$15\cdot10^{-5}$	0.0097	
α_0	0.018	0.041	0.072	0.113	0.164	0.230	
α_6	6.10^{-5}	0.0007	0.0038	0.014	0.040	0.080	
α_{12}	0	0	0	-0.0001	-0.001	-0.004	
β_2	1	1	1	1	1.002	1.037	
β_8	$15\cdot10^{-5}$	0.0017	0.0094	0.035	0.098	0.201	
β_{14}	0	0	0	0.001	0.009	0.029	

Assuming in (12) $\zeta = \mu e^{i\theta}$, we obtain the equation for the boundary of the plastic region

$$r = |w(\mu e^{i\theta})| = f(\theta)$$

In the first approximation

$$r^2 = \mu^2 [c_0 + c_1 \cos 6\theta]$$

where

$$r_{\max} = \mu \left[1 + A_6 \left(-\frac{1}{5} + \mu^6 \sum_{j=0}^{\infty} \frac{r_{3j,2}}{6j+1} \mu^{6j}\right)\right] \tag{13}$$

$$r_{\min} = \mu \left[1 + A_6 \left(\frac{1}{5} + \mu^6 \sum_{j=0}^{\infty} \frac{(-1)^j r_{3j,2}}{6j+1} \mu^{6j}\right)\right] \tag{14}$$

In Fig. 2 this boundary is represented for the case $\lambda = 0.3$, $\sigma = 2.24$ ($\mu = 0.7$, $r_{\max} = 0.730$, $r_{\min} = 0.667$).

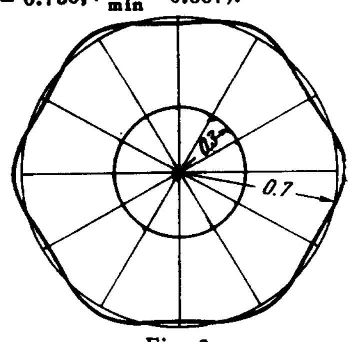

Fig. 2

Expression (14) permits to determine the smallest load for which the contour of the hole is completely encompassed by the plastic region ($r_{\min} \geq \lambda$).

Exactly in the same manner from condition $r_{\max} \leq 1$ the maximum load is determined for which plastic regions contact each other. In Fig. 3 the dependence of parameter μ on the magnitude of loading $\sigma = \sigma_0/k_0$ is given for some values of the radius of hole λ.

The effect of holes on the magnitude of the area of the plastic region s in the case of $\lambda = 0.3$ can be observed in Fig. 4. The quantity s is equal to the ratio of the area surrounded by the external contour of the plastic zone to the area of the hole $\pi\lambda^2$. In this case the curve *1* corres-

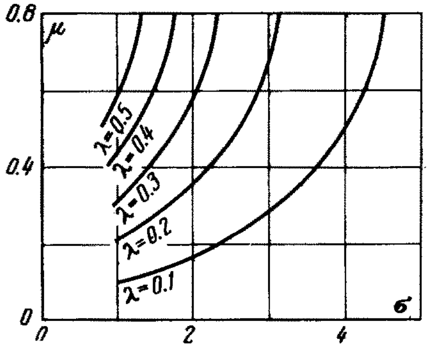

Fig. 3

ponds to the case of a single hole [4], the curve *2* to the case of a series of holes [5] and the curve *3* to the case of triangular lattice.

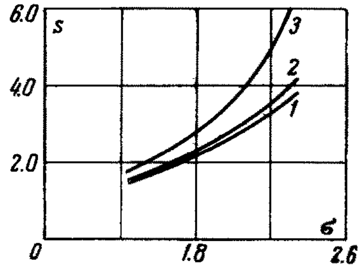

Fig. 4

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